AN INVESTIGATION OF DEFORMATIONS OF CRUCIFORM PLATES

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AN INVESTIGATION OF

DEFORMATIONS OF CRUCIFORM PLATES

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Date Approved by Chairman: August 24, 196

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LIST OF SYMBOLS

x, y	rectangular coordinates
l, d, S_j $(j=1-4)$	dimensions of the plate
h	thickness of the plate
q	intensity of uniform loading
w	transverse deflection of the plate
D	flexural rigidity of the plate
ν	Poisson's ratio
E	modulus of elasticity
M_x, M_y	bending moments per unit length of sections of plate perpendicular to x and y axes, respectively
V_x, V_y	shearing forces per unit length of sections of plate perpendicular to x and y axes, respectively

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SUMMARY

The purpose of this study is to investigate analytically the deformations of simply supported thin elastic cruciform plates under uniform loading. An experiment is performed for the confirmation of the analytical results.

The cruciform plate is considered to be a linear combination of five continuous rectangular regions, one in the center surrounded by the other four. The middle rectangular region is simply supported at its four corners with the four edges elastically connected to the four outside regions. The equation of the deformed surface is assumed to be in a series form which satisfies the natural boundary conditions and the Principle of Minimum Potential Energy.*

The four outside regions are each simply supported along their three outer edges with the remaining edge connected elastically to an edge of the middle region. For these regions, Levy's type** of solution, written in series form, is used. To solve for the unknown coefficients of the four resultant equations, the collocation method is used to match the deflections and slopes at a number of points along the common boundaries of the middle region and the outer four regions. After the coefficients of the equations of deformation have been determined, the deflections of the cruciform plate are found. A numerical example is given in which the series-form

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* See Reference 1, page 342.

** See Reference 1, page 113.

equations of deformation take only a finite number of terms. The purpose of the example is not only to show the procedure used in solving for the unknown coefficients of the equations of deformation, but also to indicate that a finite number of terms of the series will yield satisfactory results.

Experimental results are obtained for comparison with the analytical results. The steel plate, used for the experimental test, is simply supported along its edges and is uniformly loaded. The deflections of certain points of the steel plate are measured with dial gauges. Agreement between the experimental and theoretical results is satisfactory.

In this study only the case of simply supported plates with uniform loading is considered. However, the method may be extended to cases of arbitrary loading conditions and other kinds of boundary conditions, as long as the four inner corners of the cruciform plate are simply supported.

CHAPTER I

INTRODUCTION

In this study uniformly loaded cruciform plates with simply supported edges are investigated.

The problem of rectangular plates, simply supported at their corners, has been treated extensively by R. W. Hradek (2), S.L. Lee and P. Ballesteros (3), J.T. Oden (4)* and many others. A natural extension of this problem is the analysis of cruciform plates, simply supported along their edges. The cruciform plates are considered as a linear combination of five continuous rectangular regions with the central region simply supported at its four corners and elastically supported along the four edges. Cruciform plates constitute the basis for variety of plate configuration, for example, in roofing or foundation slabs.

Since this problem has had very little theoretical treatment, it is the main purpose of this study to derive the basic equations of deformations of cruciform plates, uniformly loaded and simply supported along the edges. The derivations were based on thin plate theory and the final equations were then written in series form (Chapter II).

Numerical examples are presented in Chapter III. For simplicity, only the symmetrical cruciform plate under uniform loading and

*Numbers in brackets indicate references in the Bibliography.

with simple supports along the edges is considered. The series-form equations for the deformations of the plate (See Chapter II) are applied in these examples. Since the series are convergent, it is sufficient to take only a finite number of terms of the series to obtain an approximate solution. A large number of simultaneous linear algebraic equations is involved in the determination of the coefficients of the series-form equations. These algebraic equations are solved on a digital computer. From the coefficients of these series-form equations, the deflections of the cruciform plate are obtained. The deflections at certain points of the plate are calculated at the end of Chapter III.

Chapter IV describes the tests made on a uniformly loaded thin homogeneous steel plate. The steel plate is simply supported along the eges, and the deflections of the plate at certain points are measured with dial gauges. The comparison of the theoretical and experimental results is also presented in this chapter.

CHAPTER II

DERIVATION OF BASIC EQUATIONS OF DEFORMATION OF CRUCIFORM PLATES

The equations governing the deflections of a simply supported cruciform elastic plate under uniform loading are now derived. The plate is divided into five continuous rectangular regions as shown in Fig. 1, such that regions 1,2,3 and 4 are simply supported along their outer edges with their inner edges elastically supported. Region 5 is simply supoorted only at its four corners, the other parts of its boundary are elastically connected with regions 1,2,3 and 4.

The complete analysis consists of the following parts:

Part 1: Consider region 1 (see Fig. 2) as a typical region, representative of regions 2,3 and 4. The equation of deflection for this region is obtained by Levy's method.

Part 2: Consider the center region (region 5). It is uniformly loaded rectangular plate simply supported at its four corners and elastically supported along the edges (see Fig. 3). The "Strain Energy Method" is applied to determine the equation of deflection for this region.

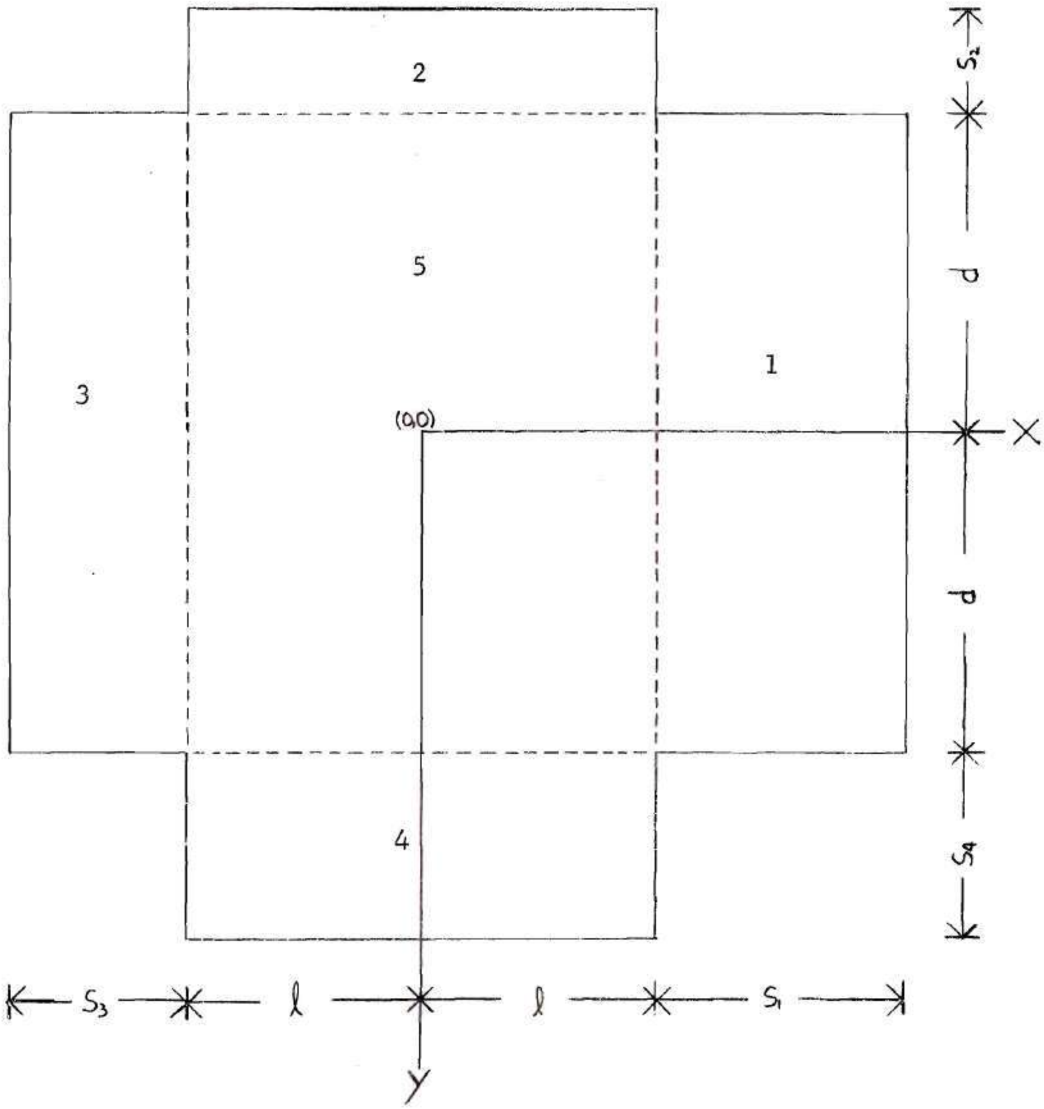
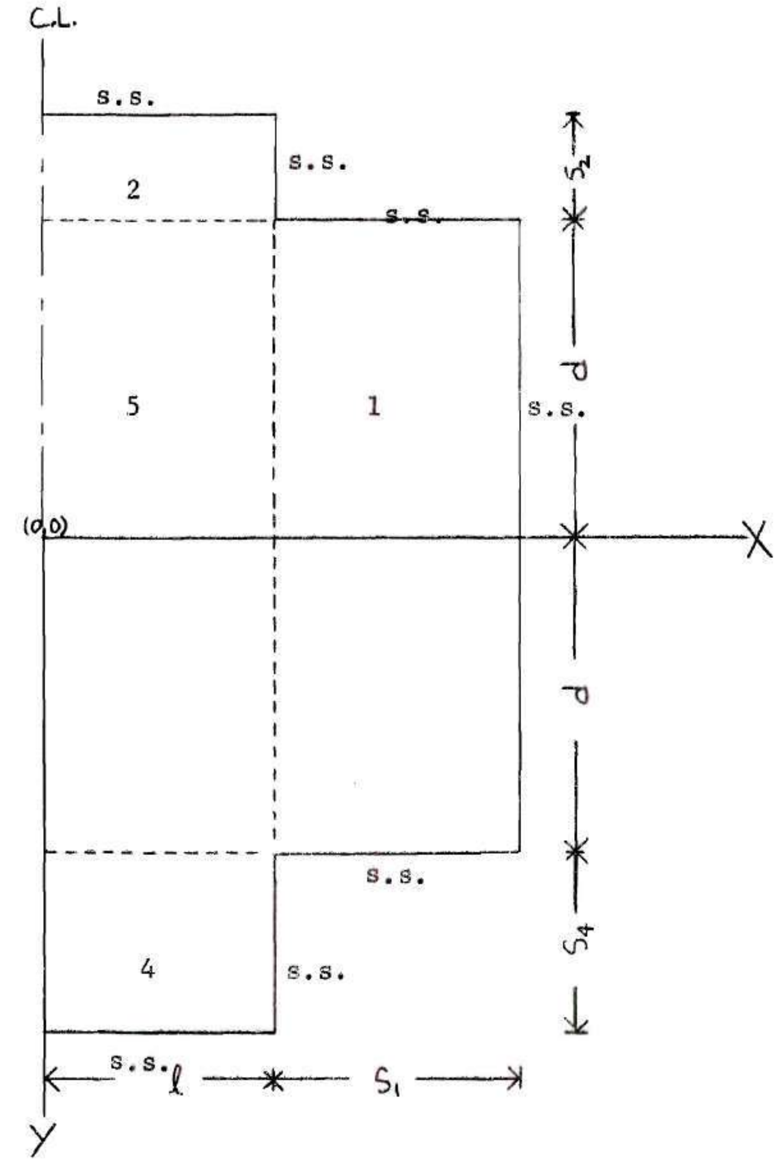


Figure 1. Coordinates of the Cruciform Plate



Note: s.s. implies simply supported.

Figure 2. Coordinates and Boundary Conditions for Region 1.

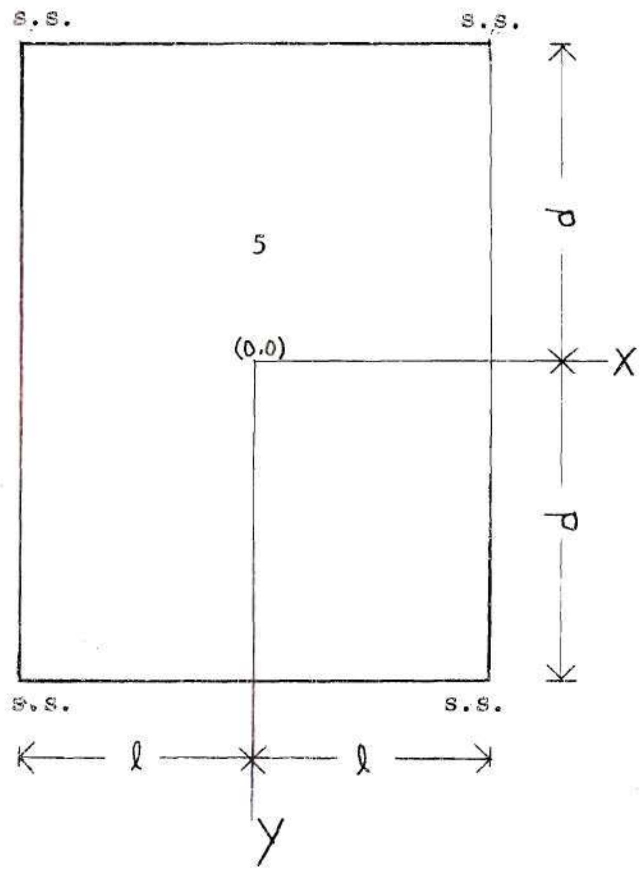


Figure 3. Coordinates and Boundary Conditions for Region 5.

Part 1. Solution of the equation of deflection for region 1, by Levy's Method:

The governing differential equation for the deflection in this case is

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}, \tag{1}$$

where

q: Intensity of uniform loading,

$D = \frac{Eh^3}{12(1-\nu^2)}$,

h: The thickness of the plate,

ν: Poisson's ratio,

w: Deflection of the plate in the direction perpendicular to xy plane.

The boundary conditions are

$w=0$ and $M_x=0$, along $x = l+s_1$

$w=0$ and $M_y=0$, along $y=d$ and $y=-d$.

At $x=l$, the deflections and the slopes of region 1 should equal the deflections and the slopes of region 5.

Assume the solution of equation (1) to be of the form

$$w_1 = w_A + w_B,$$

and take the particular solution

$$w_A = \frac{q}{24D}(y^4 - 6d^2y^2 + 5d^4),$$

where w_A represents the deflection of a uniformly loaded strip parallel to the y-axis. w_A satisfies equation (1) and the natural boundary conditions at the edges, i.e.,

$w_A = 0$ and $M_y = 0$, along $y = \pm d$.

Expansion of w_A into a Fourier series gives:

$$w_A = \frac{qd^4}{D} \sum_{m=1\cdot 3,5...}^{\infty} \left[\frac{1}{30} + 64\left(\frac{1}{m\pi}\right)^4 - \frac{8}{3}\left(\frac{1}{m\pi}\right)^2 \right] \cos b_{m1}y, \tag{2}$$

where

$$b_{m1} = \frac{m\pi}{2d} .$$

w_B, therefore, has to satisfy the homogeneous equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = 0, \tag{3}$$

and must be chosen in such a manner as to make w_1 satisfy all the boundary conditions of the plate.

Assume that w_B has the form

$$w_B = \sum_{m=1}^{\infty} X_{m1} \cos b_{m1}y, \; m=1,3,5,...,$$

Substitution of w_B into equation (3) yields

$$\sum_{m=1}^{\infty} (X_{m1}^{iv} - 2b_{m1}^2 X_{m1} + b_{m1}^4 X_{m1}) \cos b_{m1}y = 0$$

This equation can be satisfied for all values of y only if the function

X_{m1} satisfies the equation

$$X_{m1}^{iv} - 2\, b_{m1}^2\, X_{m1} + b_{m1}^4\, X_{m1} = 0$$

The general integral of this equation can be taken in the form

$$X_{m1} = 16\,\frac{qd^4}{D}\,(A_{m1}\cosh b_{m1}x + B_{m1}\, b_{m1}\, x\sinh b_{m1}x$$

$$+ C_{m1}\sinh b_{m1}x + D_{m1} b_{m1} x\cosh b_{m1}x).$$

By application of the boundary condition

$w_B = 0$ and $M_x = 0$ along $x = 1 + S_1$

C_{m1} and D_{m1} can be solved for in terms of A_{m1} and B_{m1},

$$C_{m1} = B_{m1}\left(\frac{a_{m1}}{\sinh^2 a_{m1}}\right) - A_{m1}\coth a_{m1},$$

$$D_{m1} = -B_{m1}\coth a_{m1},$$

where

$$a_{m1} = \frac{(1 + S_1)m\pi}{2d}$$

Thus, w_B becomes

$$w_B = 16\,\frac{qd^4}{D}\sum_{m=1\cdot 3}^{\infty}\Big[A_{m1}(\cosh b_{m1}x - \coth a_{m1}\sinh b_{m1}x)$$

$$+ B_{m1}(b_{m1}x\sinh b_{m1}x + \frac{a_{m1}}{\sinh^2 a_{mi}}\sinh b_{m1}x\cdot$$

$$- b_{m1}x\coth a_{m1}\cosh b_{m1}x)\Big]\cos b_{m1}y.$$

denoting

$$N_m = \left[\frac{1}{30} + 64\left(\frac{1}{m\pi}\right)^4 - \frac{8}{3}\left(\frac{1}{m\pi}\right)^2 \right] \sin\frac{m\pi}{2} \qquad ; \; m \; 1,3,5,\ldots\ldots .$$

$$I_{m1}(x) = \cosh b_{m1} x - \coth a_{m1} \sinh b_{m1} x,$$

$$J_{m1}(x) = b_{m1} x \sinh b_{m1} x + \frac{a_{m1}}{\sinh^2 a_{m1}} \sinh b_{m1} x - b_{m1} x \coth a_{m1} \cosh b_{m1} x,$$

$$K_{m1}(x) = \coth a_{m1} \cosh b_{m1} x - \sinh b_{m1} x,$$

$$L_{m1}(x) = b_{m1} x \cosh b_{m1} x + \frac{a_{m1}}{\sinh^2 a_{m1}} \cosh b_{m1} x - b_{m1} x \coth a_{m1} \cosh b_{m1} x$$

w_A and w_B become

$$w_A = \sum_{m=1\cdot 3}^{\infty} N_m \cos b_{m1} y,$$

$$w_B = 16 \frac{qd^4}{D} \sum_{m=1\cdot 3}^{\infty} \left[A_{m1} I_{m1}(x) + B_{m1} J_{m1}(x) \right] \cos b_{m1} y,$$

and the equation for the deflection of region 1, $w_1 = w_A + w_B$, becomes

$$w_1 = \frac{qd^4}{D} \sum_{m=1\cdot 3}^{\infty} \left[N_{m1} + 16 A_{m1} I_{m1}(x) + 16 B_{m1} J_{m1}(x) \right] \cos b_{m1} y, \qquad (4)$$

where

$$1 \leq x \leq 1 + S_1 \text{ and } -d \leq y \leq d.$$

Denote M_{x1}, the bending moment per unit length acting along the edge x=1 parallel to the y-axis, as

$$M_{x1} = -D\left(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2}\right)_{x=1}, \tag{5}$$

and V_{x1}, the shearing force per unit length parallel to the y-axis acting along the edge x=a, as

$$V_{x1} = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2}\right)_{x=1}. \tag{6}$$

Substitution of w_1 into equation (5) and (6) yields

$$M_{x1} = (qd^4)\sum_{m=1\cdot 3}^{\infty}\left\{\left[N_{m1} - 16\, b_{m1}^2(1-\nu)\left[A_{m1}I_{m1}(1) + B_{m1}J_{m1}(1)\right]\right.\right.$$

$$\left.\left. - 32\, b_{m1}^2 B_{m1} I_{m1}(1)\right]\right\} \cos b_{m1}y \tag{5a}$$

and

$$V_{x1} = (32qd^4)\sum_{m=1\cdot 3}^{\infty} B_{m1} b_{m1}^3 K_{m1}(1)\ \cos b_{m1}y. \tag{6a}$$

Likewise, similar equations for regions 2,3 and 4 can be derived. The equations for regions 2,3 and 4 are listed below:

Region 2: Ranges of x and y are $-1 \le x \le 1$ and $-(d+S_2) \le y \le -d$.

The equation of deflection w_2 is

$$w_2 = \frac{q1^4}{D}\sum_{m=1\cdot 3}^{\infty}\left[N_m + 16A_{m2}\ I_{m2}(y) + 16B_{m2}\ J_{m2}(y)\right] \cos b_{m2}x, \tag{7}$$

The bending moment per unit length (M_{y2}) acting along the edge y=-d, is

$$M_{y2} = (ql^4) \sum_{m=1\cdot 3}^{\infty} \left\{ \nu N_m b_{m2}^2 - 16b_{m2}^2(1-\nu) \left[A_{m1} I_{m2}(-d) + B_{m2} J_{m2}(-d) \right] - 32b_{m2}^2 B_{m2} \, I_{m2}(-d) \right\} \cos b_{m2} x. \tag{8}$$

The shearing force per unit length (V_{y2}) acting along the edge $y=-d$, is

$$V_{y2} = (32ql^4) \sum_{m=1\cdot 3}^{\infty} B_{m2} b_{m2}^3 K_{m1}(-d) \cos b_{m2} x. \tag{9}$$

Where

$$bl_{m2} = \frac{m\pi}{21}, \quad a_{m2} = \frac{-(d+S_2)m\pi}{21} \quad m=1,3,5,\ldots .$$

$$I_{m2}(y) = \cosh b_{m2} y - \coth a_{m2} \sinh b_{m2} y,$$

$$J_{m2}(y) = b_{m2} y \sinh b_{m2} y + \frac{a_{m2}}{\sinh^2 a_{m2}} \sinh b_{m2} y - b_{m2} y \coth a_{m2} \cosh b_{m2} y, \tag{9a}$$

$$K_{m2}(y) = \coth a_{m2} \cosh b_{m2} y - \sinh b_{m2} y,$$

$$L_{m2}(y) = b_{m2} y \cosh b_{m2} y + \frac{a_{m2}}{\sinh^2 a_{m2}} \cosh b_{m2} y - b_{m2} y \coth a_{m2} \sinh b_{m2} y.$$

Region 3: Ranges of x and y are -- $-(1+S_3) \le x \le -1$ and $-d \le y \le d$.

The equation of deflection w_3 is

$$w_3 = \frac{qd^4}{D} \sum_{m=1\cdot 3}^{\infty} \left[N_m + 16A_{m3} \, I_{m3}(x) + 16B_{m3} \, J_{m3}(x) \right] \cos b_{m3} y. \tag{10}$$

The bending moment per unit length (M_{x3}) acting along the edge $x=-1$, is

$$M_{x3} = (qd^4)\sum_{m=1\cdot 3}^{\infty}\left\{\nu N_m b_{m3}^2 - 16b_{m3}^2(1-\nu)\left[A_{m3}I_{m3}(-1) + B_{m3}J_{m3}(-1)\right]\right.$$

$$\left. -32b_{m3}^2 B_{m3} L_{m3}(-1)\right\} \cos b_{m3}y, \tag{11}$$

The shearing force per unit length (V_{x3}) acting along the edge $x=-1$, is

$$V_{x3} = (32qd^4)\sum_{m=1\cdot 3}^{\infty} B_{m3}b_{m3}^3 K_{m3}(-1) \cos b_{m3}y. \tag{12}$$

where

$$b_{m3} = \frac{m\pi}{2d}, \quad a_{m3} = \frac{-(1+S_3)m\pi}{2d}, \qquad m=1,3,\ldots .$$

$$I_{m3}(x) = \cosh b_{m3}x - \coth a_{m3}\sinh b_{m3}x,$$

$$J_{m3}(x) = b_{m3}x\sinh b_{m3}x + \frac{a_{m3}}{\sinh^2 a_{m3}}\sinh b_{m3}x - b_{m3}x\coth a_{m3}\cosh b_{m3}x,$$

$$K_{m3}(x) = \coth b_{m3}\cosh b_{m3}x - \sinh b_{m3}x, \tag{12a}$$

$$L_{m3}(x) = b_{m3}x\cosh b_{m3}x + \frac{a_{m3}}{\sinh^2 a_{m3}}\cosh b_{m3}x - b_{m3}x\coth a_{m3}\sinh b_{m3}x.$$

Region 4: Ranges of x and y-- $-1\le x\le 1$ and $-d\le y\le d$.

The equation of deflection w_4 is

$$w_4 = \frac{ql^4}{D}\sum_{m=1\cdot 3}^{\infty}\left[N_m + 16A_{m4}I_{m4}(y) + 16B_{m4}J_{m4}(y)\right] \cos b_{m4}x. \tag{13}$$

The bending moment per unit length (M_{y4}) acting along the edge $y=d$, is

$$M_{y4} = (ql^4)\sum_{m=1\cdot 3}^{\infty}\left\{\nu N_m b_{m4}^2 - 16b_{m4}^2(1-\nu)\left[A_{m4}I_{m4}(d) + B_{m4}J_{m4}(d)\right]\right.$$

$$\left. - 32b_{m4}^2 B_{m4} I_{m4}(d)\right\} \cos b_{m4}x. \tag{14}$$

The shearing force per unit length (V_{y4}) acting along the edge y=d, is

$$V_{y4} = (32ql^4)\sum_{m=1\cdot 3}^{\infty} B_{m4} b_{m4}^3 K_{m4}(d) \cos b_{m4}x. \tag{15}$$

where

$$b_{m4} = \frac{m\pi}{2l}, \quad a_{m4} = \frac{m\pi(d + S4)}{2l}, \ m=1,3,\ldots .$$

$$I_{m4}(y) = \cosh b_{m4}v - \coth a_{m4} \sinh b_{m4}y,$$

$$J_{m4}(y) = b_{m4}y \sinh b_{m4}y + \frac{a_{m4}}{\sinh^2 a_{m4}} \sinh b_{m4}y - b_{m4}y \coth a_{m4} \cosh b_{m4}y,$$

$$K_{m4}(y) = \coth a_{m4} \cosh b_{m4}y - \sinh b_{m4}y, \tag{15a}$$

$$L_{m4}(y) = b_{m4}y \cosh b_{m4}y + \frac{a_{m4}}{\sinh^2 a_{m4}} \cosh b_{m4}y - b_{m4}y \coth a_{m4} \sinh b_{m4}y.$$

Part 2. Derivation of the equation of deflection for region 5 by the Strain Energy Method:

Assume the equation of deflection for this region to be of the form

$$w_5 = \sum_{n=1}^{\infty} H_n \left[2 - \left(\frac{x}{l}\right)^{2n} - \left(\frac{y}{d}\right)^{2n}\right], \tag{16}$$

which satisfies the boundary conditions

$$w_5(\pm 1, \pm d) = 0,$$

$$\frac{\partial w_5}{\partial x}(0,0) = 0, \qquad \frac{\partial w_5}{\partial y}(0,0) = 0,$$

and

$$w_5 \neq 0; \quad \frac{\partial w_x}{\partial x} \neq 0 \qquad \text{along } x = \pm 1,$$

$$w_5 \neq 0; \quad \frac{\partial w_5}{\partial y} \neq 0 \qquad \text{along } y = \pm d.$$

The properties of equation (16) are:

(a) The maximum value of w_5 is at x=0 and y=0.

(b) w_5 is a uniformly convergent series:

At x=1 and y=0, $w_1 = w_5$, therefore,

$$\sum_{n=1}^{\infty} H_n = \frac{qd^4}{D} \sum_{m=1 \cdot 3}^{\infty} \left[N_m + 16A_{m1} I_{m1}(1) + 16B_m J_{m1}(1) \right], \qquad (16a)$$

and since the right hand side of equation (16a) is known to be a convergent series, so

$\sum_{n=1}^{\infty} H_n$ converges, and

$\sum_{n=1}^{\infty} 2H_n$ also converges.

Since

$\sum_{n=1}^{\infty} 2H_n$ converges,

and

$$H_n \left[2 - (\tfrac{x}{l})^{2n} - (\tfrac{y}{d})^{2n}\right] \leq 2 H_n, \text{ for all n and every x and y}$$

in the region $-l \leq x \leq l$, $-d \leq y \leq d$.

therefore,

$$\sum_{n=1}^{\infty} H_n \left[2 - (\tfrac{x}{l})^{2n} - (\tfrac{y}{d})^{2n}\right]$$

converges uniformly according to the Weierstrass' M-test.*

(c) w_5 is continuous and the 2nth partial derivatives of w_5 with respect to x and y exist.

The expression for the total strain energy, denoted by V, of the plate is

$$V = -\frac{D}{2}\int_x\int_y \left\{\left(\frac{\partial^2 w_5}{\partial x^2} + \frac{\partial^2 w_5}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w_5}{\partial x^2} \quad \frac{\partial^2 w_5}{\partial y^2}\right] - \left(\frac{\partial^2 w_5}{\partial x \partial y}\right)^2\right\} dxdy.$$

The total potential energy of the external load, represented by the symbol Q, is

$$Q = -\int_{-l}^{l}\int_{-d}^{d} q w_5 dxdy.$$

The energy due to moments acting along the edges, denoted as R, is

$$R = -\int_{-d}^{d} M_{x1}\left(\frac{\partial w_1}{\partial x}\right)_{x=l} dy - \int_{-l}^{l} M_{y2}\left(\frac{\partial w_2}{\partial y}\right)_{y=d} dx - \int_{-d}^{d} M_{x3}\left(\frac{\partial w_3}{\partial x}\right)_{x=-l} dy$$

$$-\int_{-l}^{l} M_{x4}\left(\frac{\partial w_4}{\partial y}\right)_{y=d} dx.$$

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*See Reference 11, page 396.

The energy due to shear forces acting along the edges, denoted as 5, is

$$S = -\int_{-d}^{d} V_{x1}w_1 \,_{x=1}\, dy - \int_{-1}^{1} V_{y2}w_2)_{y=-d}dx - \int_{-d}^{d} V_{x3}w_3)_{x=-1}dy - \int_{-1}^{1} V_{y4}w_4)_{y=d}dx.$$

Therefore, the total energy, denoted as P, of region 5 becomes

$$p = V + Q + R + S.$$

The Principle of Minimum Potential Energy requires that $\frac{\partial D}{\partial H_i} = 0,$ thus,

$$\frac{\partial P}{\partial H_i} = \frac{\partial V}{\partial H_i} + \frac{\partial Q}{\partial H_i} + \frac{\partial R}{\partial H_i} + \frac{\partial S}{\partial H_i} = 0. \qquad (i=1,2,\ldots\ldots,n.)$$

Since w_5 converges uniformly, the integrands of the expressions for V, Q, R and S are also uniformly convergent and are continuous. Therefore, the operations of integration and differentiation are interchangeable and for fixed i, (i=1,2,...,n.) the partial derivative of the total strain energy V with respect to H_i become .

$$\frac{\partial V}{\partial H_i} = \frac{D}{2}\int_{-1}^{1}\int_{-d}^{d} \frac{\partial}{\partial H_i} \left\{ \sum_{n=1}^{\infty}\left[H_n(2n)(2n-1)(\tfrac{1}{1})^{2n}\, x^{2n-2}\right]^2 \right.$$
$$\sum_{n=1}^{\infty}\left[H_n(2n)(2n-1)(\tfrac{1}{d})^{2n}\, y^{2n-2}\right]^2$$
$$\left. -2\nu\left[\sum_{n=1}^{\infty} H_n(2n)(2n-1)(\tfrac{1}{1})^{2n}\, y^{2n-2}\right]\left[\sum_{n=1}^{\infty} H_n(2n)(2n-1)(\tfrac{1}{d})^{2n} y^{2n-2}\right]\right\} dxdy$$
$$= 4D \sum_{n=1}^{\infty} H_n(2n)(2n-1)(2i)(2i-1)\left[\frac{1}{2(n+i)-3}\left(\frac{1^2\, d^2}{1^2 d^2}\right)\right]$$

(equation concluded on following page)

$$+\sum_{n=1}^{\infty} \nu H_n(2n)(2i) \quad \frac{1^{4n} + d^{4n}}{1^{2n+1} \; d^{2n+1}}\Bigg\} \qquad (17)$$

The potential energy due to uniform loading is

$$Q = -\int_{-1}^{1}\int_{-d}^{d} q\sum_{n-1}^{\infty} H_n\left[2 - (\tfrac{x}{1})^{2n} - (\tfrac{x}{d})^{2n}\right] dxdy,$$

and

$$\frac{\partial Q}{\partial H_2} = -16q1d(\frac{i}{2i\;1}) \; . \qquad i=1,2,\ldots,n. \qquad (18)$$

The energy due to moments acting along the edges is

$$R = -(qd^4)\int_{-d}^{d}\left[\sum_{n=1}^{\infty}(2n)H_n\right]\Bigg\{\sum_{m=1\cdot 3}^{\infty}\Big[\nu N_m - 16b_{m1}^2(1-\nu)(A_{m1}I_{m1}(1) + B_{m1}J_{m1}(1)$$

$$-32b_{m1}^2B_{m1}I_{m1}(1)\Big]\cos b_{m1}y + \sum_{m=1\cdot 3}^{\infty}\Big[\nu N_m - 16b_{m3}^2(1-\nu)(A_{m3}J_{m3}(-1)+B_{m3}J_{m3}(-1)$$

$$-32b_{m3}^2B_{m3}I_{m3}(-1)\Big] \cos b_{m1}y\Bigg\} - (q1^4)\int_{-1}^{1}\sum_{n=1}^{\infty}\left[(2n)H_n\right]\Bigg\{\sum_{m=1\cdot 3}^{\infty}\Big[\nu N_m$$

$$-16b_{m2}^2(1-\nu)(A_{m2}I_{m2}(-d) + B_{m2}J_{m2}(-d))-32b_{m2}^2B_{m2}I_{m2}(-d)\Big]\cos b_{m2}x+\sum_{m=1\cdot 3}^{\infty}\Big[\nu N_m$$

$$-16b_{m4}^2(1-\nu)(A_{m4}I_{m4}(d)+B_{m4}J_{m4}(d)-32b_{m4}^2B_{m4}I_{m4}(d)\Big] \quad \cos b_{m4}x\Bigg\}$$

therefore, the partial derivatives of R with respect to H_A, $i=1, 2,..,n.$, are

$$\frac{\partial R}{\partial H_i} = -(4q)(2i)\sum_{m=1\cdot 3}^{\infty}\left\{\left[\nu N_m - 16b_{m1}^2(1-\nu)(A_{m1}I_{m1}(1)+B_{m1}J_{m1}(1))\right.\right.$$
$$\left. -32b_{m1}^2 B_{m1} I_{m1}(1)\right]\left(\frac{1}{b_{m1}d}\right)\sin b_{m1}d + \left[\nu N_m - 16b_{m3}^2\ (1-\nu)(A_{m3}I_{m3}(-1)\ B_{m3}J_{m3}(-1))\right.$$
$$\left. -32b_{m3}^2 B_{m3} I_{m3}(-1)\right]\left(\frac{1}{b_{m3}d}\right)\sin b_{m3}d + \left[\nu N_m - 16b_{m2}(1-\nu)(A_{m2}I_{m2}(-d)\ B_{m2}J_{m2}(-d))\right.$$
$$\left. -32b_{m2}^2 B_{m2} I_{m2}(-d)\right]\left(\frac{1}{b_{m2}1}\right)\sin b_{m2}1 + \left[\nu N_m - 16b_{m4}(1-\nu)(A_{m4}I_{m4}(d)\ B_{m4}J_{m4}(d))\right.$$
$$\left.\left. -32b_{m4}^2 B_{m4} I_{m4}(d)\right]\left(\frac{1}{b_{m4}1}\right)\sin b_{m4}1\right\}. \tag{19}$$

The energy due to shear forces acting along the edges is

$$S = -\int_{-d}^{d}\left\{\sum_{n=1} H_n\left[1-\left(\frac{y}{d}\right)\right]^{2n}\right\}\left[\sum_{m=1\cdot 3}^{\infty} 32qd^4\right]\left\{\left[b_{m1}^3 K_{m1}(1) B_{m1}\cos b_{m1}y\right]\right.$$
$$\left. +\left[b_{m3}^3 K_{m3}(-1) B_{m3}\cos b_{m3}y\right]\right\} dy - \int_{-1}^{1}\sum_{n=1} H_n\left[1-\left(\frac{x}{1}\right)\right]^{2n}\left\{\left[\sum_{m=1\cdot 3}^{\infty} 32q1^4\right]\right.$$
$$\left.\left[b_{m2}^3 K_{m2}(-d) B_{m2}\cos b_{m2}x\right] + \left[b_{m4}^3 K_{m4}(d) B_{m4}\cos b_{m4}x\right]\right\} dx,$$

and the partial derivatives of v with respect to H_i (i=1,2,..,n.) are

$$\frac{\partial S}{\partial H_i} = -\sum_{m=1\cdot 3}^{\infty} 32(qd^4)\left\{\left[b_{m1}^2 K_{m1}(1) B_{m1}(2\sin b_{m1}d - d^{-2i}X_{im}^{(1)})\right.\right.$$
$$\left.\left. +\left[b_{m3}^2 K_{m3}(-1) B_{m3}(2\sin b_{m3}d - d^{-2i}X_{im}^{(3)})\right]\right\}$$
$$-\sum_{m=1\cdot 3}^{\infty} 32(q1^4)\left\{\left[b_{m2}^2 K_{m2}(-d) B_{m2}(2\sin b_{m2}1 - 1^{-2i}X_{im}^{(2)})\right]\right.$$
$$\left. +\left[b_{m4}^2 K_{m4}(d) B_{m4}(2\sin b_{m4}1 - 1^{-2i}X_{im}^{(4)})\right]\right\}, \tag{20}$$

where $X_{im}^{(1)}$, $X_{im}^{(2)}$, $X_{im}^{(3)}$ and $X_{im}^{(4)}$ are given as

$$X_{im}^{(1)} = b_{m1}\int_{-d}^{d} y^{2i}\cos b_{m1}y\ dy,$$

$$X_{im}^{(2)} = b_{m2}\int_{-1}^{1} x^{2i}\cos b_{m2}x\ dx,$$

$$X_{im}^{(3)} = b_{m3}\int_{-d}^{d} y^{2i}\cos b_{m3}y\ dy,$$

$$X_{im}^{(4)} = b_{m4}\int_{-1}^{1} x^{2i}\cos b_{m4}x\ dx. \tag{20a}$$

Substitution of equations (17), (18), (19) and (20) into

$$\frac{\partial P}{\partial H_i} = \frac{\partial V}{\partial H_i} + \frac{\partial Q}{\partial H_i} + \frac{\partial R}{\partial H_i} + \frac{\partial S}{\partial H_i} = 0,$$

results in the following system of linear algebraic equations:

$$\begin{aligned} D\sum_{n=1}^{\infty} g_{in}H_n = {} & 2c_i q l d + 2d^2 \sum_{m=1\cdot 3}^{\infty} (e_{im}^{(1)}A_{m1} + f_{im}^{(1)}B_{m1} - h_{im}^{(1)}) \\ & + 2d^2 \sum_{m=1\cdot 3}^{\infty} (e_{im}^{(3)}A_{m3} + f_{im}^{(3)}B_{m3} - h_{im}^{(3)}) \\ & + 2l^2 \sum_{m=1\cdot 3}^{\infty} (e_{im}^{(2)}A_{m2} + f_{im}^{(2)}B_{m2} - h_{im}^{(2)}) \\ & + 2l^2 \sum_{m=1\cdot 3}^{\infty} (e_{im}^{(4)}A_{m4} + f_{im}^{(4)}B_{m4} - h_{im}^{(4)}), \end{aligned} \tag{21}$$

where i varies from 1 to n. Thus, the system of equations (21) has 8m n unknowns (H_1, H_2,....H_n, A_{1j}, A_{2j},....,A_{mj} and B_{1j}, B_{2j},....B_{mj}, j=1, 2, 3, & 4) and n equations, where

$$g_{in} = (2n)(2n-1)(2i)(2i-1)\left[\frac{1}{(n\ I)-3}\ \frac{(1^2+d^2)}{(1^2d^2\ )}\right] + \nu\ (2n)(2i)\frac{(1^{4n}\ d^{4n}\ \ )}{(1^{2n}\ 1_d 2n\ 1)},$$

$$c_i = \frac{i}{2i+1},$$

$$e_{im}^{(1)} = 8i(db_{m1})(1-\nu)I_{m1}(1)\ \sin b_{m1}d,$$

$$e_{im}^{(2)} = 8i(1b_{m2})(1-\nu)I_{m2}(-d)\ \sin b_{m2}1,$$

$$e_{im}^{(3)} = 8i(db_{m3})(1-\nu)I_{m3}(-1)\ \sin b_{m3}d,$$

$$e_{im}^{(4)} = 8i(1b_{m4})(1-\nu)I_{m4}(d)\ \sin b_{m4}1,$$

$$f_{im}^{(1)} = (4d^2b_{m1}^2K_{m1}(1))(2\sin b_{m1}d-d^{2i}X_{im}^{(1)})-8i(db_{m1})(1-\nu)J_{m1}(d)\sin b_{m1}d+2\ e_{im}^{(1)},$$

$$f_{im}^{(2)} = (41^2b_{m2}^2K_{m2}(-d))(2\sin b_{m2}1-1^{2i}X_{im}^{(2)})-8i(1b_{m2})(1-\nu)J_{m2}(-d)\sin b_{m2}1+2\ e_{im}^{(2)},$$

$$f_{im}^{(3)} = (4d^2b_{m3}^2K_{m3}(-1))(2\sin b_{m3}d-d^{2i}X_{im}^{(3)})-8i(db_{m3})(1-\nu)J_{m3}(-1)\sin b_{m3}d+2\ e_{im}^{(3)},$$

$$f_{im}^{(4)} = (41^2b_{m4}^2K_{m4}(d))(2\sin b_{m4}1-1^{2i}X_{im}^{(4)})-8i(1b_{m4})(1-\nu)J_{m4}(d)\sin b_{m4}1+2\ e_{im}^{(4)},$$

$$h_{im}^{(j)} = \frac{2i}{db_{mj}}\nu N_m\ \sin b_{mj}d, \quad j=1 \text{ and } 3,$$

$$h_{im}^{(j)} = \frac{2i}{1b_{mj}}\nu N_m\ \sin b_{mj}1, \quad j=2 \text{ and } 4. \tag{21a}$$

The following equations are obtained from the boundary conditions:

1. Along $x=1$

(a) $W_1 = W_5$:

$$\sum_{n=1}^{\infty} H_n\left[1-\left(\frac{y}{d}\right)\right]^{2n} = \sum_{m=1\cdot 3}^{\infty} \frac{qd^4}{D}\left[N_m+16A_{m1}I_{m1}(1)+16B_{m1}J_{m1}(1)\right]\cos b_{m1}y, \tag{22}$$

(b) $\frac{\partial w_1}{\partial x} \quad \frac{\partial w_5}{\partial x}$:

$$-\sum_{n=1}^{\infty} nH_n \sum_{m=1\cdot 3}^{\infty} \frac{8qd^4}{D}(b_{m1}d)\left[A_m K_{m1}(1)+B_{m1}(L_{m1}(1)-K_{m1}(1))\right]\cos b_{m1}y. \tag{23}$$

2. Along $x=-1$,

(a) $w_3 = w_5$:

$$\sum_{n=1}^{\infty} H_n\left[1-(\frac{y}{d})\right]^{2n} = \sum_{m=1\cdot 3}^{\infty} \frac{qd^4}{D}\left[N_m+16A_{m3}I_{m3}(-1)+16B_{m3}J_{m3}(-1)\right]\cos b_{m3}y, \tag{24}$$

(b) $\frac{\partial w_3}{\partial x}=\frac{\partial w_5}{\partial x}$:

$$-\sum_{n=1}^{\infty} nH_n = \sum_{m=1\cdot 3}^{\infty} \frac{8qd^4}{D}(b_{m3}d)\left[A_{m3}K_{m3}(-1)+B_{m3}(L_{m3}(-1)-K_{m3}(-1))\right]\cos b_{m3}y. \tag{25}$$

3. Along $y=-d$,

(a) $w_2 = w_5$:

$$\sum_{n=1}^{\infty} H_n\left[1-(\frac{x}{1})\right]^{2n} = \sum_{m=1\cdot 3}^{\infty} \frac{q1^4}{D}\left[N_m+16A_{m2}I_{m2}(-d)+16B_{m2}J_{m2}(-d)\right]\cos b_{m2}x, \tag{26}$$

(b) $\frac{\partial w_2}{\partial y}=\frac{\partial w_5}{\partial y}$:

$$-\sum_{n=1}^{\infty} nH_n = \sum_{m=1\cdot 3}^{\infty} \frac{8q1^4}{D}(b_{m2}a)\left[A_{m2}K_{m2}(-d)+B_{m2}(L_{m2}(-d)-K_{m2}(-d))\right]\cos b_{m2}x, \tag{27}$$

4. Along $y=d$,

(a) $w_4 = w_5$:

$$\sum_{n=1}^{\infty} H_n\left[1-(\frac{x}{1})\,2n\right] = \sum_{m=1\cdot 3}^{\infty} \frac{q1^4}{D}\left[N_m+16A_{m4}I_{m4}(d)+16B_{m4}J_{m4}(d)\right]\cos b_{m4}x, \tag{28}$$

(b) $\frac{\partial w_4}{\partial y} = \frac{\partial w_5}{\partial y}$:

$$-\sum_{n=1}^{\infty} nH_n = \sum_{m=1\cdot 3}^{\infty} \frac{8ql^4}{D}(b_{m4}l)\left[A_{m4}K_{m4}(d)+B_{m4}(L_{m4}(d)-K_{m4}(d))\right]\cos b_{m4}x. \tag{29}$$

Equations (22) to (29), together with y, y_1, y_2,...,y_m., give 8m equation. Therefore, the 8m n unknowns ($H_1, H_2, \ldots H_n$, $A_{1j}, A_{2j}, \ldots A_{mj}$, and B_{1j}, B_{2j},...,B_{mj}, j=1,2,3 and 4) can be obtained from the system of equations (21).

Since w_5 and w_j (j=1,2,3, and 4) are convergent series, a finite number of H_n, A_{mj} and B_{mj} (j=1,2,3, and 4) are sufficient to determine the equations of deflection for regions 1, 2, 3, 4, and 5. Thus, a finite number of points along the common boundaries will be taken and the boundary conditions will be satisfied exactly at these points. Obviously the inclusion of more points will yield greater accuracy in the results. Once the coefficients (H_n, A_{mj} and B_{mj}; j=1,2,3,4) of the equations of deflection for regions (1,2,3,4 and 5) are found, the deflections of the cruciform plate are determined.

As an illustration, two numerical examples are presented in Chapter III.

CHAPTER III

NUMERICAL EXAMPLES

Two numerical examples are presented in this Chapter.

The first example considers the matching of one point on the common boundary and the second example considers that of three points on the common boundary. For simplicity a symmetrical cruciform plate (Fig. 4) under uniform loading is considered.

From equation (15) in Chapter II, the equation of deflection for the middle region (region 5) is

$$w_5=\sum_{n=1}^{\infty} H_n\left[2-\left(\frac{N}{1}\right)^{2n}-\left(\frac{y}{1}\right)^{2n}\right], \tag{30}$$

and equation of deflection for region 1 (see equation (4) in Chapter II) is

$$w_1=\frac{ql^4}{D}\sum_{m=1\cdot 3}^{\infty}\left[N_m+16A_{m1}I_{m1}(x)+16B_{m1}J_{m1}(x)\right]\cos b_{m1}y.$$

Due to symmetry the deflections of corresponding points of regions 1,2,3 and 4 are the same, therefore, it is sufficient to consider region as a typical one for region 2,3, and 4.

If the general expression for the deflections of regions 1,2,3 and 4, is denoted as w, then

$$w=\frac{ql4}{D}\sum_{m=1\cdot 3}^{\infty}\left[N_m+16A_mI_m(x)+16B_mJ_m(x)\right]\cos b_{m1}y. \tag{31}$$

In view of equation (3a) in Chapter II,

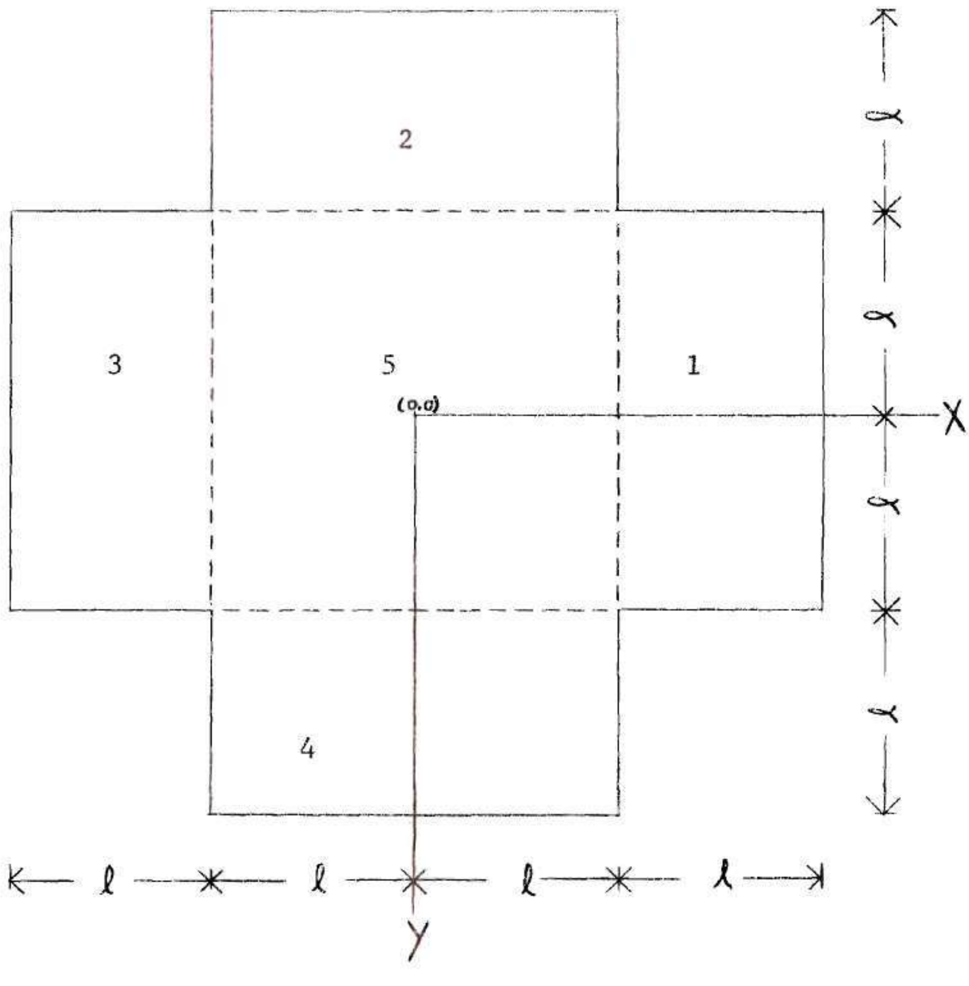

Figure 4. Coordinates for the Symmetrical

Cruciform Plate.

$$a_m = m\pi \ , \ b_m = \frac{m\pi}{2l} \ ,$$

$$N_m = \left[\frac{1}{30} - 64\left(\frac{1}{m\pi}\right)^4 - \frac{8}{3}\left(\frac{1}{m\pi}\right)^2\right] \sin\frac{m\pi}{2}, \quad m=1,3,\ldots .$$

$$I_m(x) = \cosh b_m x - \coth a_m \sinh b_m x,$$

$$J_m(x) = b_m x \sinh b_m x + \frac{a_m}{\sinh^2 a_m} \sinh b_m x - b_m x \coth a_m \cosh b_m x, \tag{31a}$$

$$K_m(x) = \coth a_m \cosh b_m x - \sinh b_m x,$$

$$L_m(x) = b_m x \cosh b_m x + \frac{a_m}{\sinh^2 a_m} \cosh b_m x - b_m x \coth a_m \sinh b_m x.$$

The partial derivatives of the total energy of the cruciform plate with respect to H_i $(i=1,2,\ldots,n.)$, as obtained from the system of equations (21), become

$$D \sum_{n=1}^{\infty} g_{in} H_n = 2c_i q l^2 + 8l^2 \sum_{m=1\cdot 3}^{\infty} (e_{im} A_m + f_{im} B_m - h_{im}), \tag{32}$$

where, from equation (21a), the coefficients are given as below:

$$g_{in} = (2n)(2i)\left(\frac{1}{l}\right)^2 \left[\frac{(2n-1)(2i-1)}{2(n\ i)-3} + \nu\right],$$

$$c_i = \frac{2i}{2i+1} \ ,$$

$$e_{im} = 8i(lb_m)(1-\nu) I_m(l) \sinh b_m l,$$

$$f_{im} = (4lb_m K_m(l))(2\sinh b_m l - l^{-2i} X_{im}) - 8i(lb_m)(1-\nu) J_m(l) \sinh b_m l + 2e_{im},$$

$$X_{im} = b_m \int_{-1}^{1} y^{2i} \cos b_m y \, dy,$$

$$h_{im} = \frac{2i}{b_m l} \nu N_m \sinh b_m l. \tag{32a}$$

The boundary conditions along $x=1$ are

$$w_1 = w_5 \text{ and } \frac{\partial w_1}{\partial x} = \frac{\partial w_5}{\partial x} .$$

Thus, from equation (22), the equation resulting from matching deflections along the common boundary of regions 1 and 5 becomes

$$\sum_{n=1}^{\infty} H_n \left[1-\left(\frac{y}{1}\right)\right]^{2n} = \frac{q1^4}{D} \sum_{m=1\cdot 3}^{\infty} \left[N_m + 16A_m I_m(1) + 16B_m J_m(1)\right] \cos b_m y. \tag{33}$$

From equation (23), the equation resulting from matching slopes along the common boundary of regions 1 and 5 becomes

$$-\sum_{n=1}^{\infty} nH_n = \frac{8q1^4}{D} \sum_{m=1\cdot 3}^{\infty} (b_m 1)\left[A_m K_{m4}(1) + B_m(L_m(1) - K_m(1))\right] \cos b_m y. \tag{34}$$

Therefore, the equations of deflection for region 1 (equation (31)), and for region 5 (equation (30)) follow from the solutions of the linear algebraic equations (32),(33) and (34).

In the following examples, equations (32),(33) and (34) are applied, and the coefficients in equations (31a) and (32a) are determined. Poisson's ratio is taken to be 0.3.

Example 1

Matching of one point at $x=1$ and $y=0$ on the common boundary of region 1 and of region 5 with $n=1,2.$:

When taking $n=1,2.$, there are two unknowns (H_1, H_2) obtained from the equation of deflection for region 5. As equation (30) indicates,

$$w = H_1 \left[2-\left(\frac{x}{1}\right)^2 - \left(\frac{y}{1}\right)^2\right] + H_2\left[2-\left(\frac{x}{1}\right)^4 - \left(\frac{y}{1}\right)^4\right]. \tag{35}$$

From equation (32), since $i=1,2.$, there are two linear algebraic equations. By taking one point at $x=1$ and $y=0$, on the common boundary there are only two linear algebraic equations obtained (see equation (33) and (34)). Thus, only two more unknowns can be determined in this example. If m=1 is chosen, two unknowns results, namely A_1 and B_1.

The equation of deflection for region 1 from equation (31) becomes

$$w=\frac{ql^4}{D}\left[N_1+16A_1I_1(x)+16B_1J_1(x)\right]\cos b_1y, \tag{36}$$

and from equation (32), the partial derivative of total energy of the plate with respect to H_1 is

$$g_{11}H_1+g_{12}H_2=\frac{ql^4}{D}(c_1+e_{11}A_1+f_{11}B_1-h_{11}), \tag{37}$$

and the partial derivative of total energy w.r.t. H_2 is

$$g_{21}H_1+g_{22}H_2=\frac{ql^4}{D}(c_2+e_{21}A_1+f_{21}B_1-h_{21}). \tag{38}$$

At x=1, y=0 (since m=1) equation (33), matching deflections of region 1 and deflections of region 5, becomes

$$H_1+H_2=\frac{ql^4}{D}\left[N_1+16A_1I_1(1)+16B_1J_1(1)\right], \tag{39}$$

and equation (34), matching slopes of region 1 and region 5, becomes

$$-(H_1+2H_2)=\frac{8ql^4}{D}(lb_1)\left[A_1K_1(1)+B_1(L_1(1)-K_1(1))\right]. \tag{40}$$

Equations (35) and (36) are the equations of deflection for the cruciform plate; equations (37),(38),(39) and (40) are the governing linear algebraic equations from which the unknowns (H_1,H_2,A_1 and B_1) can be determined.

The coefficients of equations (39), (40) are given by equations (31a) and (32a) and are listed below:

$$\begin{aligned} &N_1= 0.21, && K_1(1)= 0.2171,\\ &I_1(1)= 0.1997, && L_1(1)= 0.3772.\\ &J_1(1)= -0.2825, && \end{aligned} \tag{40a}$$

$$g_{11}=5.2\frac{1}{l^2},\quad g_{12}=10.4\frac{1}{l^2},\quad g_{21}=10.4\frac{1}{l^2},\quad g_{22}=33.6\frac{1}{l^2}.$$

$$c_1=0.667,\quad c_2=0.8.$$

$$X_{11}=0.377,\quad X_{12}=0.162. \tag{40b}$$

$$e_{11}=7.023,\quad e_{12}=14.046.$$

$$f_{11}=46.927,\quad f_{12}=77.432.$$

$$h_{11}=0.187,\quad h_{12}=0.374.$$

Substitution of equation (40a) into (35) and (36) results in

$$H_2=-\frac{ql^4}{D}\,(1.283B_1+0.0417), \tag{41}$$

and
$$H_1=\frac{ql^4}{D}\,(1.351A_1+11.59B_1+0.175) \tag{42}$$

Substitution of equation (40b) into (39) and (40) gives

$$H_1+H_2=\frac{ql^4}{D}\left[0.21+16(0.2A_1-0.283B_1)\right], \tag{43}$$

and
$$-(H_1+2H_2)=\frac{ql^4}{D}\left[12.56(0.217A_1-0.16B_1)\right]\quad. \tag{44}$$

Thus, A_1, B_1, H_1 and H_2, as determined from equations (41),(42),(43) and (44), are

$$A_1=-0.02596,\qquad B_1=0.00189,$$

$$H_1=0.162\quad,\qquad H_2=-0.044\ .$$

The equations of deflection (equations (35) and (36)) for region 1 and region 5, respectively, are

$$w_5=\frac{ql^4}{D}\left\{0.162\left[2-\left(\frac{x}{l}\right)^2-\left(\frac{y}{l}\right)^2\right]^2-0.044\left[2-\left(\frac{x}{l}\right)^4-\left(\frac{y}{l}\right)^4\right]\right\}, \tag{45}$$

and
$$w_1=\frac{ql^4}{D}\left[0.21+16(0.00189I_1(x)-0.026J_1(x))\right]\cos b_1y. \tag{46}$$

At points shown in Fig. 5, the deflections are calculated as shown in the following table.

Table 1. Deflections of the Plate in Example 1

Points	Location of point		$w/ql^4/D$
	x	y	
1	0	0	0.2358
2	1/2	0	0.1981
3	1	0	0.1179
4	1/2	1/2	0.1603
5	1	1/2	0.0802
6	31/2	0	0.1028
7	31/2	1/2	0.0624

Example 2

Matching of deflections and slopes of region 1 and region 5 at three points ((1,0),(1,1/2) and (1,1/4)) on the common boundary with n=1,2,..,6.:

The equation of deflection for region 5 (see equation (30)), now becomes

$$w_5 = \sum_{n=1}^{\infty} H_n \left[2-\left(\frac{x}{l}\right)^{2n} - \left(\frac{y}{l}\right)^{2n} \right]. \tag{47}$$

From equation (32), for n=1,2,..,6., the partial derivatives of the total energy of the plate with respect to the H_i's are

$$D\sum_{n=1}^{\infty} g_{in}H_n = 2c_i ql^4 + 8l^2 \sum_{m=1\cdot 3}^{\infty} (e_{im}A_m + f_{im}B_m - h_{im}).\quad i=1,2,..,6. \tag{32'}$$

Since i varies from 1 to 6, six linear algebraic equations are obtained.

Along the common boundary (x=1) of region 1 and region 5, take $y=y_j$, j=1,2,3. ($y_1=0$, $y_2=1/2$, $y_3=1/4$). Equation (33), matching deflections of region 1 and region 5, becomes

$$\sum_{n=1}^{\infty} H_n \left[1-\left(\frac{y_j}{l}\right)\right]^{2n} = \frac{ql^4}{D} \sum_{m=1\cdot 3}^{\infty} \left[N_1 + 16A_m I_m(1) + 16B_m J_m(1)\right] \cos b_m y_j, \tag{33'}$$

and equation (34), matching slopes of region 1 and region 5, becomes

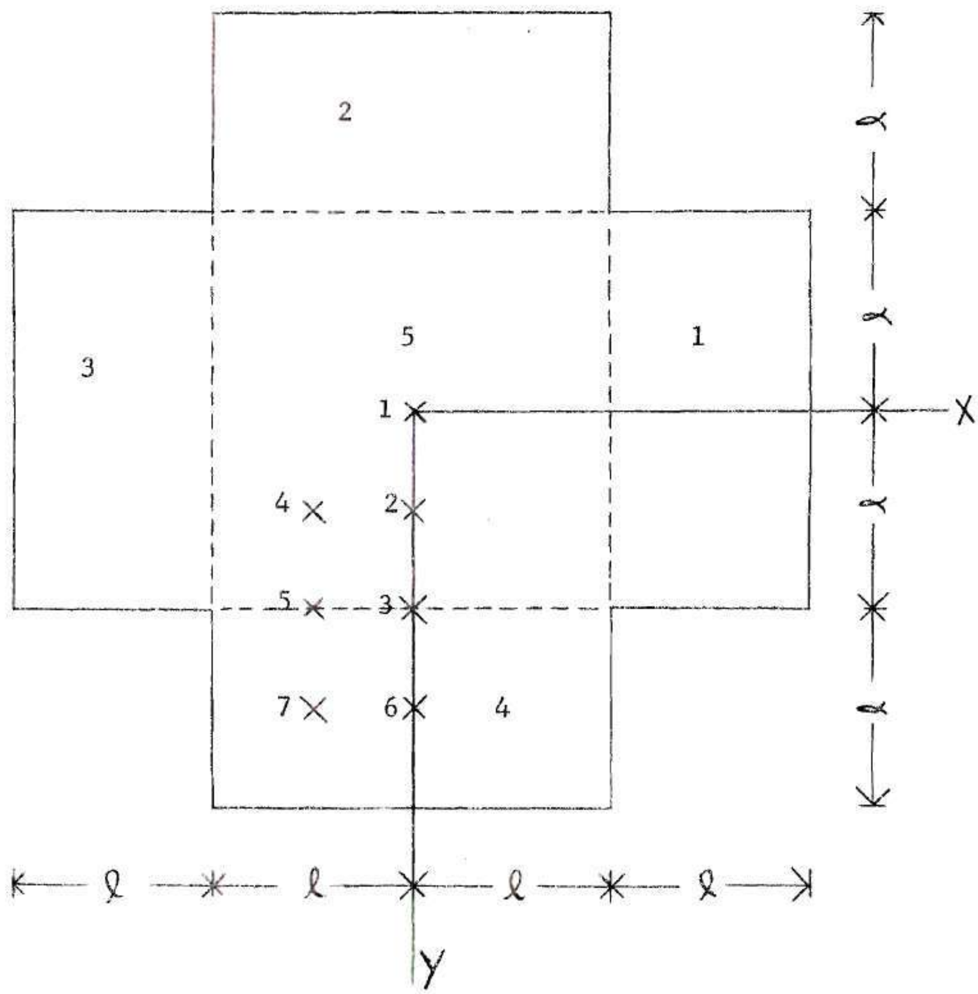

Figure 5. Locations of the Data Points.

$$-\sum_{n=1}^{\infty} nH_n = \frac{ql^4}{D} \sum_{m=1\cdot 3}^{\infty} (b_m l)\left[A_m K_m(1) + B_m(K_m(1) - L_m(1))\right] \cos b_m y_j. \tag{34'}$$

Six linear algebraic equations are obtained from equations (33') and (34'). By letting m=1,3,5., six more unknowns (A_j and B_j, j=1,2,3.) result. Thus, one finally obtains a system of twelve equations in twelve unknowns.

For m=1,3,5., the equation of deflection for region 1 (see equation (31)) becomes

$$w = \frac{ql^4}{D} \sum_{m=1\cdot 3}^{\infty} \left[N_m - 16A_m I_m(x) - 16B_m J_m(x)\right] \cos b_m y. \tag{48}$$

When taking n=1,2,..,6., i=1,2,..,6. and m=1,3,5., the coefficients in equations (33'),(34') and (32') are given by equations (31a) and (32a):

From equation (31a):

m = 1	m = 3	m = 5	
$N_1 = 0.21$,	$N_3 = -0.0043$,	$N_5 = 0$,	
$I_1(1)=0.1997$,	$I_3(1)=0.009$,	$I_5(1)=0$,	
$J_1(1)=-0.2825$,	$J_3(1)=-0.0433$,	$J_5(1)=-0.016$,	(48a)
$K_1(1)=0.2171$,	$K_3(1)=0.009$,	$K_5(1)=0$,	
$L_1(1)=0.3772$.	$L_3(1)=0.0433$.	$L_5(1)=0.016$.	

From equation (32a), writing the coefficients in matrix form,

$$l^2\left[g_{ij}\right] = \begin{bmatrix} 5.2 & 10.4 & 15.6 & 20.8 & 26 & 31.2 \\ 10.4 & 33.6 & 58.628 & 84.267 & 110.182 & 134.246 \\ 15.6 & 58.628 & 110.8 & 167.127 & 225.692 & 285.6 \\ 20.8 & 84.267 & 167.127 & 260.431 & 360.0 & 463.624 \\ 26.0 & 110.182 & 225.692 & 360.0 & 506.471 & 625.263 \\ 31.2 & 136.246 & 285.6 & 463.624 & 625.263 & 872.914 \end{bmatrix},$$

$$\left[c_i\right] = \left[0.667 \quad 0.8 \quad 0.857 \quad 0.889 \quad 0.909 \quad 0.923\right],$$

For m=1, i=1,2,..,6.,

$$\left[e_{i1}\right] = \left[\ 7.023 \quad 14.046 \quad 21.069 \quad 28.092 \quad 35.115 \quad 42.138\right],$$

$$\left[f_{i1}\right] = \left[46.927 \quad 77.432 \quad 113.148 \quad 121.893 \quad 699.387 \quad -28850.615\right],$$

$$\left[h_{i1}\right] = \left[\ 0.187 \quad 0.374 \quad 0.561 \quad 0.748 \quad 0.934 \quad 1.121\right].$$

For m= 3; i= 1,2,..,6.,

$$\left[e_{i3}\right] = \left[-0.95 \quad -1.899 \quad -2.849 \quad -3.798 \quad -4.748 \quad -5.697\right],$$

$$\left[f_{i3}\right] = \left[-9.619 \quad -20.492 \quad -31.519 \quad -41.227 \quad -50.647 \quad -60.0\right],$$

$$\left[h_{i3}\right] = \left[\ 0.000055 \quad 0.0001099 \quad 0.000165 \quad 0.000275 \quad 0.0003295\right].$$

For m=5; n= 1,2,..,6.,

$$\left[e_{i5}\right] = \left[h_{i5}\right] = \left[0\right],$$

$$\left[h_{i5}\right] = \left[2.763 \quad 2.092 \quad 0.827 \quad 0.7008 \quad 0.6966 \quad -0.839\right].$$

Therefore, the partial derivatives of the total energy with respect to the H_i's, i=1,2,..,6., obtained from equation (32') becomes

$$\left[g_{ij}\right]\left[H_j\right] = \frac{ql^4}{D}\left[c_i + \sum_{m=1\cdot 3}^{\infty} \left(e_{im}A_m + f_{im}B_m - h_{im}\right)\right]. \tag{49}$$

The system of equations (49) includes 6 equations.

Equation (33'), matching deflections of region 1 and region 5 at x=1, becomes

(1) $y_1 = 0$

$$H_1 + H_2 + H_3 + H_4 + H_5 + H_6 = \frac{ql^4}{D}(-3.195A_1 - 0.213A_3 - 0.144A_5 + 4.52B_1$$

$$+ 0.678B_3 + 0.651B_5) = 0.2279\frac{ql^4}{D}, \tag{50}$$

(2) $y_2 = 1/2$

$$0.75H_1 + 0.9375H_2 + 0.984H_3 + 0.9961H_4 + 0.999H_5 + 0.9998H_6$$

$$+ \frac{ql^4}{D}\left(-2.26A_1 - 0.102A_3 + 3.1965B_1 - 0.4796B_3\right) = 0.1485\frac{ql^4}{D}, \tag{51}$$

(3) $y_3 = 1/4$

$$0.9375H_1+0.9961H_2+0.9998H_3+0.99998H_4+0.999999H_5$$
$$+0.9999999H_6+\frac{ql^4}{D}\left[-2.95A_1-0.55A_3-0.15A_5+4.18B_1+0.26B_3\right.$$
$$\left.+0.46B_5\right]=0.224\frac{ql^4}{D} \qquad (52)$$

Equation (34'), matching slopes of region 1 and region 5 at x=1, becomes

(1) $y_1 = 0$

$$H_1+2H_2+3H_3+4H_4+5H_5+6H_6+\frac{ql^4}{D}\left[-2.727A_1+0.339A_3+0.334A_5\right.$$
$$\left.-2.01B_1-1.26B_3-0.74B_5\right]=0, \qquad (53)$$

(2) $y_2 = 1/2$

$$H_1+2H_2+3H_3+4H_4+5H_5+6H_6+\frac{ql^4}{D}\left[1.93A_1-0.24A_3-1.42B_1\right.$$
$$\left.-0.8896B_3\right]=0. \qquad (54)$$

(3) $y_3 = 1/4$

$$H_1+2H_2+3H_3+4H_4+5H_5+6H_6+\frac{ql^4}{D}\left[2.52A_1-0.13A_3-0.24A_5-1.86B_1\right.$$
$$\left.-0.48B_3-0.523B_5\right]=0. \qquad (55)$$

The system of equations (49) and equations (50),...,(55), are solved on an electronic digital computer* (Burroughs 220) resulting in the following values of the unknowns:

$$H_1=-0.468\ \frac{ql^4}{D},\qquad H_2=3.065\ \frac{ql^4}{D},\qquad H_3=-5.335\ \frac{ql^4}{D},$$
$$H_4=3.916\ \frac{ql^4}{D},\qquad H_5=-1.129\ \frac{ql^4}{D},\qquad H_6=0.0499\ \frac{ql^4}{D},$$

---

*See Appendix for the computer program (Page 46).

$$A_1 = -0.056, \qquad A_3 = 0.0911, \qquad A_5 = 1.809,$$

$$B_1 = -0.0015, \qquad B_3 = 0.121, \qquad B_5 = 0.418.$$

Hence, the equations of deflection for region 1 and region 5 are determined:

$$w_5 = \frac{ql^4}{D}\left\{-0.468\left[2-\left(\frac{x}{l}\right)^2-\left(\frac{y}{l}\right)^2\right]+3.065\left[2-\left(\frac{x}{l}\right)^4-\left(\frac{y}{l}\right)^4\right]\right.$$
$$-5.335\left[2-\left(\frac{x}{l}\right)^6-\left(\frac{y}{l}\right)^6\right]+3.916\left[2-\left(\frac{N}{l}\right)^8-\left(\frac{y}{l}\right)^8\right]$$
$$\left.-1.129\left[2-\left(\frac{x}{l}\right)^{10}-\left(\frac{y}{l}\right)^{10}\right]+0.0499\left[2-\left(\frac{x}{l}\right)^{12}-\left(\frac{y}{l}\right)^{12}\right]\right\}.$$

$$w = \frac{ql^4}{D}\left\{\left[N_1-0.869I_1(x)-0.024J_1(x)\right]\cos\frac{\pi}{2}y\right.$$
$$+\left[N_3+1.4576I_3(x)+1.936J_3(x)\right]\cos\frac{3\pi}{2}y$$
$$\left.+\left[N_5+28.944I_5(x)+6.688J_5(x)\right]\cos\frac{5\pi}{2}y\right\}.$$

For the points shown in Fig. 5, the deflections are calculated as shown in the following table.

Table 2. Deflections of the Plate in Example 2

Points	Location of point x	y	$w/ql^4/D$
1	0	0	0.199
2	1/2	0	0.1934
3	1	0	0.10
4	1/2	1/2	0.187
5	1	1/2	0.0934
6	3 1/2	0	0.0912
7	3 1/2	1/2	0.0727

CHAPTER IV

EXPERIMENTAL RESULTS

An experiment has been performed to determine the deflections of a simply supported homogeneous steel plate under uniform loading.

A symmetric cruciform steel plate, 0.049 inch thick, is used for the specimen. The dimensions of the steel plate are shown in Fig. 6.

The steel plate is simply supported by knife-edged steel strips which are attached to the inner side of a wooden structure (see Fig. 7 and Fig. 8). A uniform load of 0.1937 pounds per square inch (see Fig. 9) is placed on the plate. The corners of the steel plate are fastened so that they are in contact with the supports at all times during the process of loading. The deflections of the plate are measured at the indicated points (see Fig. 10) by using dial gauges. Table 3 presents the results of these measurements.

Table 3. Measured Deflections of the Steel Plate

Point	1	2	3	4	5	6	7
in inch	0.209	0.199	0.123	0.18	0.11	0.10	0.0702
w/w_o*	1	0.952	0.588	0.861	0.526	0.479	0.336

*w_o is the deflection at point 1.

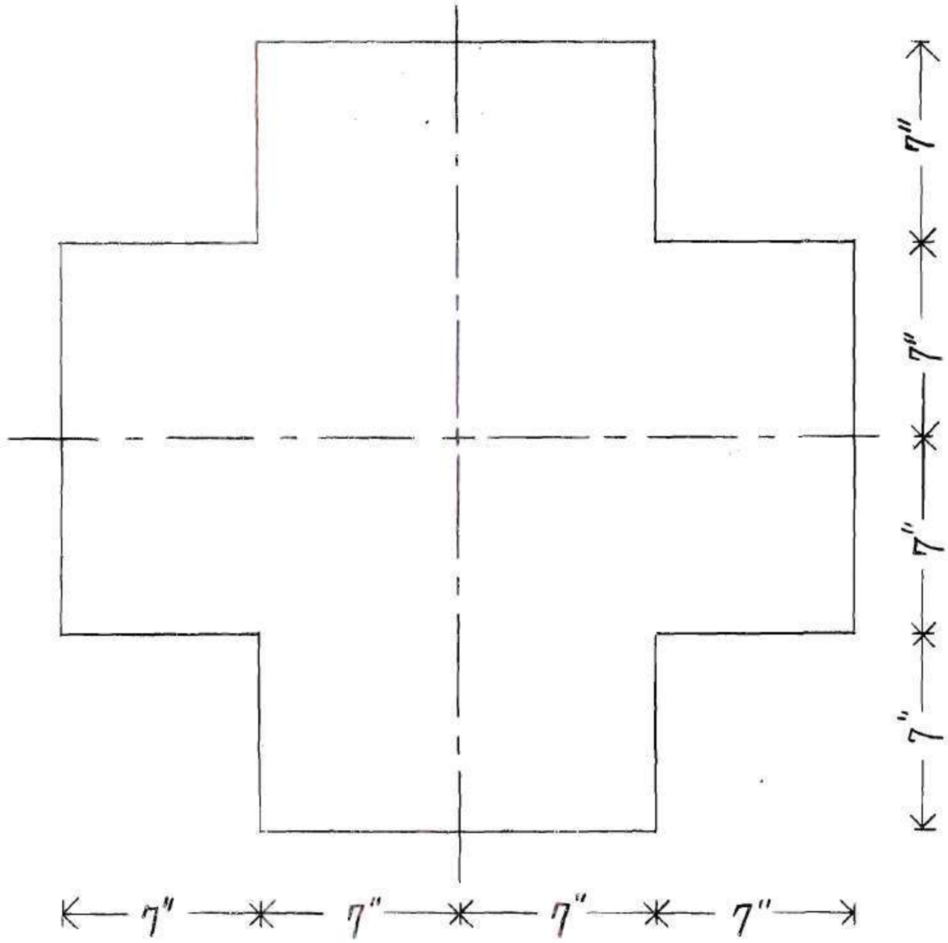

Figure 6. The Dimensions of the Steel Plate

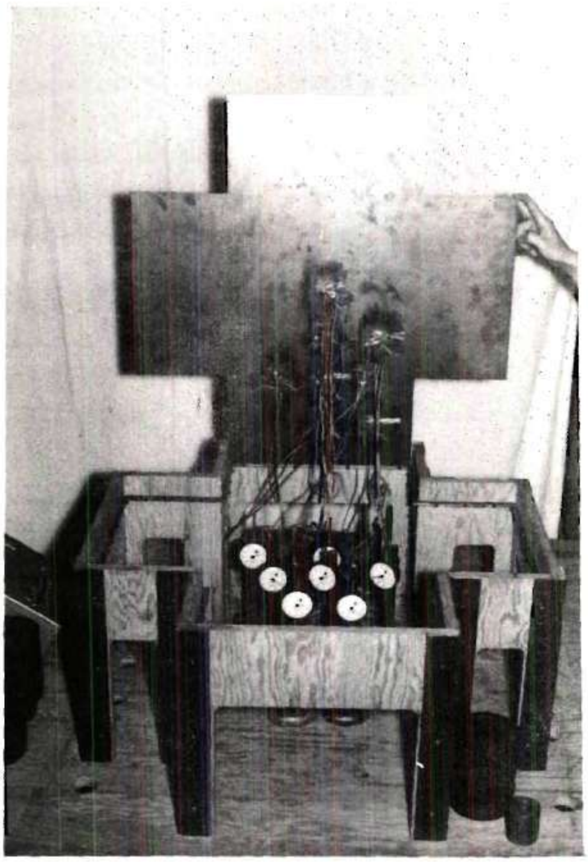

Figure 7. Experimental Set Up

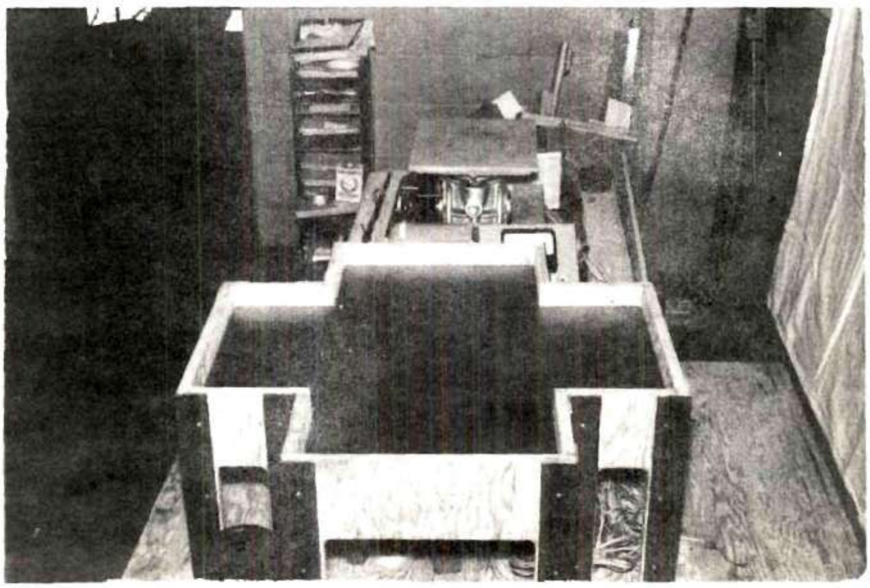

Figure 8. Steel Plate Simply Supported by the Wooden Structure.

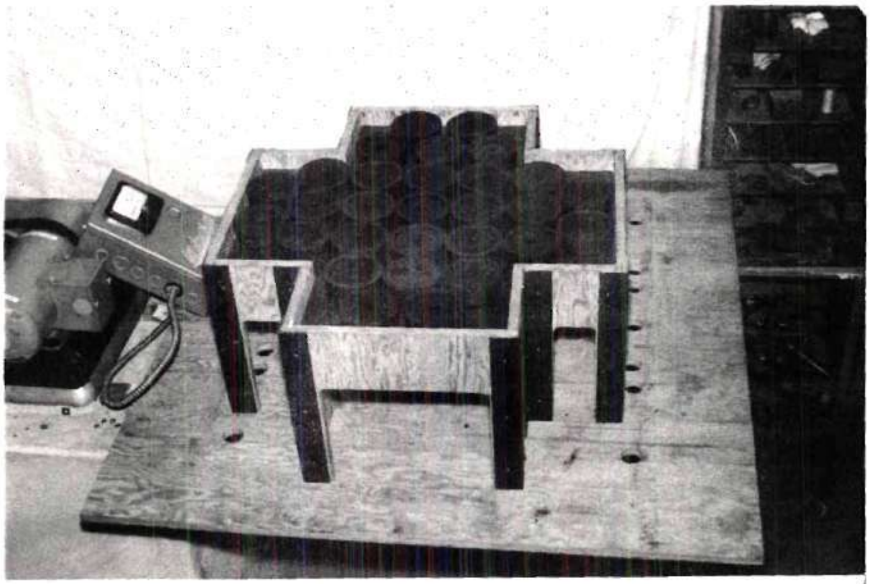

Figure 9. Uniform Loading on the Steel Plate.

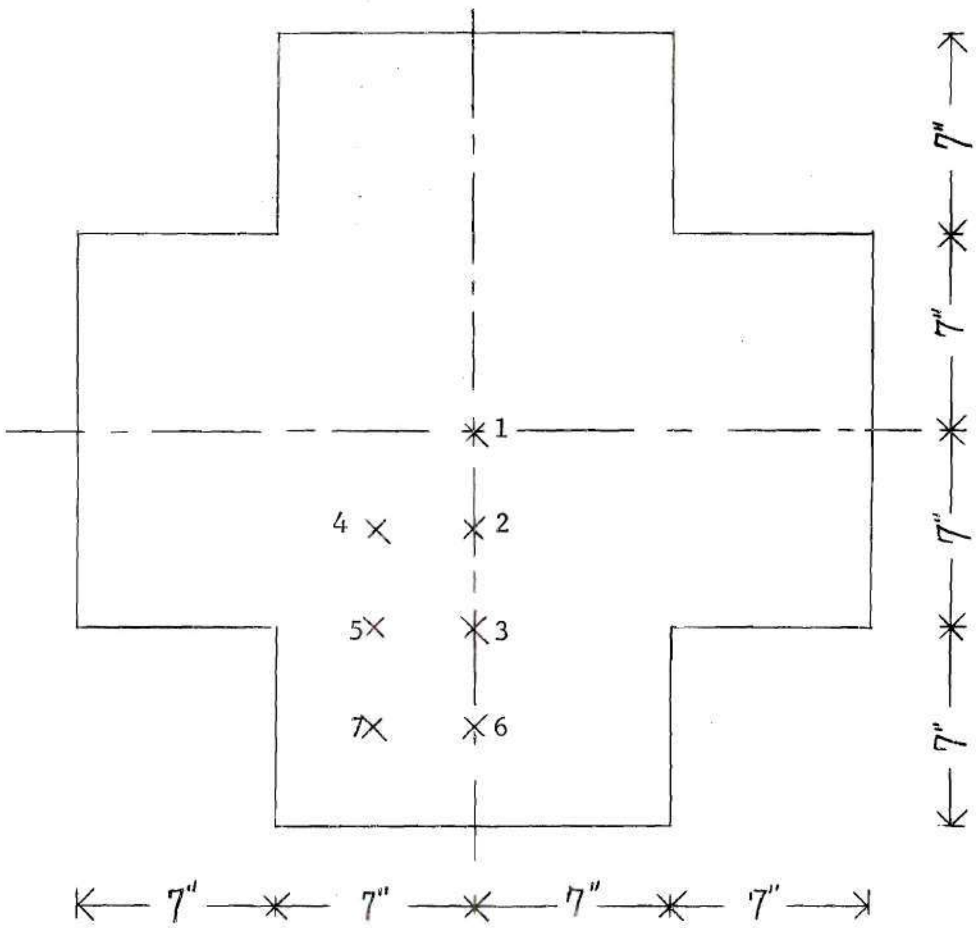

Figure 10. Points on the Plate Measured by Using Dial Gauges.

The comparison of theoretical and experimental results is based on the following data:

Total Area of the Steel Plate $A = 588$ in.

$q = 0.1937\ \mathrm{lb/in^2}$

$h = 0.049$ in.

$\nu = 0.3$

$E = 30\mathrm{x}10^6$ psi.

$D = \frac{Eh^3}{12(1-\nu^2)} = 323.21$ lb/in.

and

$\frac{ql^4}{D} = 1.466$ in.

Thus, from Table 1 and Table 2, the deflections at the points shown in Fig. 10 are obtained:

Points		1	2	3	4	5	6	7
w/w_o*	Ex.1	1	0.84	0.50	0.68	0.340	0.436	0.265
	Ex.2	1	0.972	0.503	0.94	0.469	0.458	0.365

The comparison between the theoretical and experimental results is shown in Table 4 and Fig. 11.

Table 4. Comparison of Theoretical and Experimental Results

Point		1	2	3	4	5	6	7
w/w_o	Ex.1	1	0.84	0.50	0.68	0.34	0.436	0.265
	Ex.2	1	0.972	0.503	0.94	0.469	0.458	0.365
	Test	1	0.952	0.588	0.861	0.526	0.479	0.336

*w_o is the deflection at point 1.

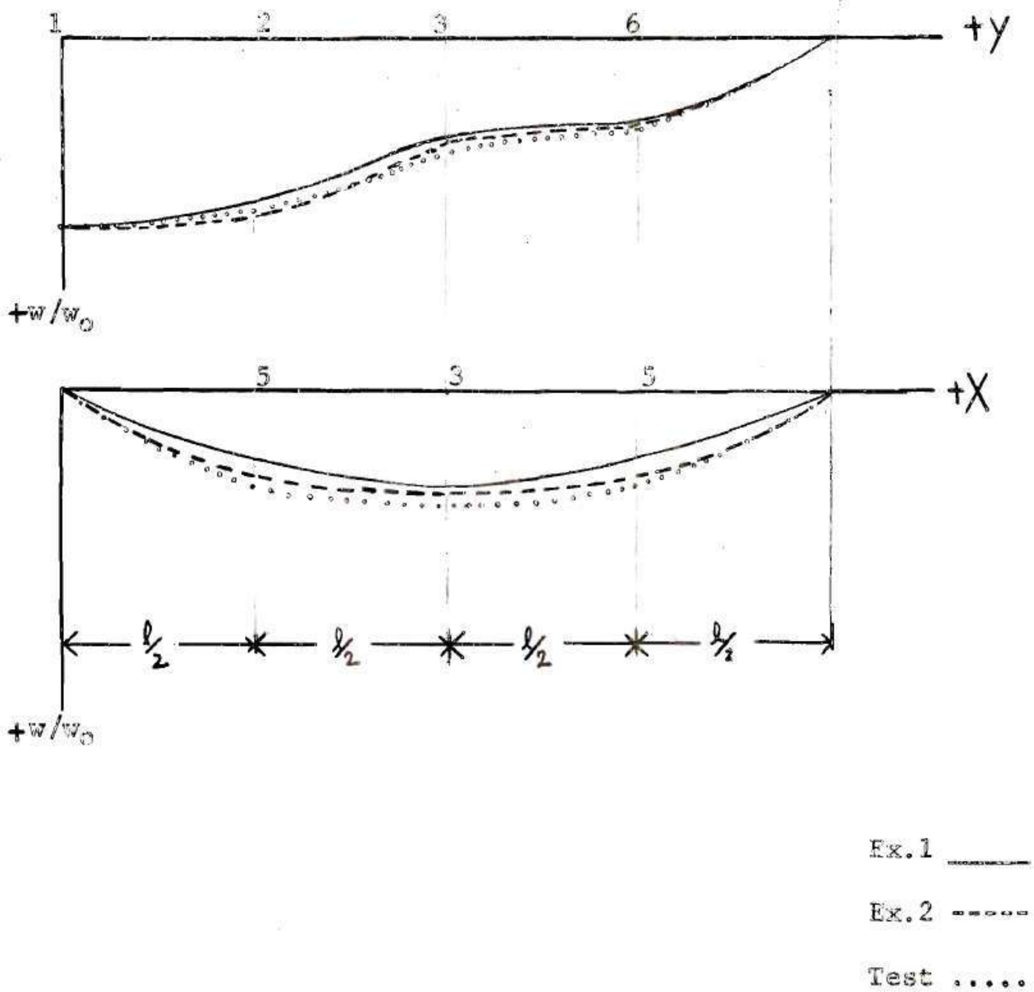

Figure 11. Comparison of Normalized Deflection Curve of the Theoretical and Experimental Results

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

The theoretical and experimental results shown in Table 4 and Fig. 11 indicate satisfactory agreement. The maximum deviation in deflections is approximately 14 per cent.

The discrepancies may be explained by consideration of the following factors:

1. Only a few points were used to match conditions along the common boundaries, since the number of unknowns increased at the rate of four times the number of points used, and the resultant system of equations becomes cumbersome for present considerations.
2. The supporting device in the experiment does not simulate exactly the condition of a simply supported edge.
3. The loading technique does not result in a truly uniform load.

The investigation discussed in this study considers only simply supported cruciform plates under uniform load. However, the general method and derivations presented in this study can be extended to other types of supporting conditions as well as other loading conditions.

The author wishes to make the following recommendations for further investigations:

(1). Application of the technique presented in this thesis to non-uniformly loaded cruciform plates with various supporting conditions.

(2). Application to plates which consist of various combinations of rectangular regions other than the cruciform shape.

(3) Additional experimental investigations including measurements of strains as well as of displacements.

APPENDIX

```
0200              BAC-220 STANDARD VERSION    2/1/62
0200           COMMENT SIMULTANEOUS EQUATIONS                          $
0200           INTEGER M,N,I,J,EQN                                     $
0200           ARRAY (A(12,13), X(12,1))                               $
0200      PROCEDURE JORDAN (N,M,A(,)$DET,X(,))                         $  GJ   1
0200      BEGIN                                                           GJ   2
0200      COMMENT THIS PROCEDURE SOLVES A SET OF M LINEAR SYSTEMS OF N    GJ   4
0200      EQUATIONS AND N UNKNOWNS  A1(N,N).X(N,M) = A2(N,M) WITH COMMON MATRIX  GJ   5
0200       A1 APPLYING JORDANS METHOD WITH SEARCHING FOR PIVOTS. AT THE SAME     GJ   6
0200      TIME THE ABS VALUE OF THE DETERMINANT IS COMPUTETD A =(A1/A2)  $  GJ   7
0200      INTEGER N,M,K,L,J,Y,Z                                        $  GJ   8
0204           DET = 1                                                 $  GJ   9
0206           FOR K = (1,1,N)                                         $  GJ  10
0217      BEGIN    D = 0                                               $  GJ  11
0219           L = K                                                   $  GJ  12
0221           FOR J = (K,1,N)                                         $  GJ  13
0232      BEGIN    IF ABS(A(J,K)) GTR D                                $  GJ  14
0232      BEGIN    L = J                                               $  GJ  15
```

```
0247            D = ABS(A(L,K))                                    END  END       $  GJ  16
0257            IF L NEQ K                                                        $  GJ  17
0261    BEGIN   FOR J = (K,1,N+M)                                                 $  GJ  18
0275    BEGIN   D = A(L,J)                                                        $  GJ  19
0285            A(L,J) = A(K,J)                                                   $  GJ  20
0301            A(K,J) = D                                         END END        $  GJ  21
0311            DET = DET . A(K,K)                                                $  GJ  22
0321    EITHER IF ABS(DET) GTR 1**-8                                              $  GJ  23
0328    BEGIN   FOR Y = (K+1,1,N)                                                 $  GJ  24
0338    BEGIN   D = A(Y,K)/A(K,K)                                                 $  GJ  25
0357            FOR Z = (K+1,1,N+M)                                               $  GJ  26
0372            A(Y,Z) = A(Y,Z) - A(K,Z). D                        END END        $  GJ  27
0400    OTHERWISE $ GO TO ILLCOND                          $ 1..   END            $  GJ  28
0403            DET = ABS(DET)                                                    $  GJ  29
0405            FOR K = (1,1,M)                                                   $  GJ  30
0416            FOR J = (N,-1,1)                                                  $  GJ  31
0428    BEGIN   D = A(J,K+N)                                                      $  GJ  32
0439            FOR L = (J+1,1,N)                                                 $  GJ  33
```

```
0451            D = D - A(J,L).X(L,K)                                     $  GJ  34
0471            X(J,K) = D/A(J,J)                        END              $  GJ  35
0491     RETURN                                                           $  GJ  36
0493     ILLCOND..WRITE ($$ILL)                                           $  GJ  37
0497            STOP    $ RETURN                                          $  GJ  38
0500     FORMAT  ILL(W0, *MATRIX IS TOO BADLY ILL-CONDITIONED FOR THIS PROCEDURE  GJ  39
0500     TO INVERT.*,W0)                                                     GJ  40
0517                                             END JORDAN()     $  GJ  41
0544            EON = 1                                                   $
0546            N = 12                                                    $
0548            M = 1                                                     $
0550            INPUT DATA(FOR I = (1,1,12)                               $
0563                 (FOR J = (1,1,13)                                    $
0575            A(I,J)))                                                  $
0589     PSS1..  READ($$ DATA)                                            $
0593            WRITE ($$ R,RR)                                           $
0601            WRITE ($$ S,SS)                                           $
0609            EON = EON + 1                                             $
```

```
0612            IF EQN EQL 3  $  EQN = EQN + 2                          $
0620            JORDAN (N,M,A( , )$DET,X( , ))                          $
0643            WRITE ($$ T,TT)                                         $
0651            IF EQN LSS 2  $  GO TO PSS1                             $
0657            STOP                                                    $
0658    OUTPUT  (R(EQN),
0665            S(FOR I = (1,1,12)                                      $
0678            (FOR J = (1,1,13)                                       $
0690            A(I,J))),
0704            T(FOR I = (1,1,12) $ (I,X(I,1))))  $
0728    FORMAT  RR(B14,*EQUATION SET*,W4,
0728            (B14,I3,W0,)),
0728            SS(W6,
0728            (B18,*COEFFICIENTS*,W0,
0728            (F14.8,B2,F14.8,B2,F14.8,W0,
0728            (F14.8,B2,F14.8,B2,F14.8,W0,)))),
0728            TT(W4,
0728            (B25,*G*,I3,* = *,F14.8,W0))  $
```

```
0770                  FINISH                                                      $
COMPILED PROGRAM ENDS AT 0771
PROGRAM VARIABLES BEGIN AT 4351
```

EQUATION SET
1

COEFFICIENTS

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.45205840, 01	.65088000, 00	.67824000, 00
.22794860, 00	.75000000, 00	.93750000, 00
.98434500, 00	.99609380, 00	.99902340, 00
.99975590, 00	-.22596257, 01	.00000000, 00
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.40000000, 01	.50000000, 01	.60000000, 01
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G 3 = -.53351294, 01
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G 5 = -.11287076, 01
G 6 = .49818906,-01
G 7 = -.55991682,-01
G 8 = .18094592, 01

```
G  9 =  .91072777,-01
G 10 = -.14993410,-02
G 11 =  .41757067, 00
G 12 =  .12107006, 00
```

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